Filters with Decreased Passband Error

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***Abstract*—Most analog low-pass filters consist of poles. Stable, causal all-pole filters give 20dB/dec asymptotic error $|1-H(s)|$ in the pass band. Pole-zero filters are typically used to improve performance around crossover, and shift error between phase and gain, but common causal filter types still give 20dB/dec asymptotic error in the passband. This paper describes the theory and practice of a class of filters which can have arbitrarily good asymptotic behavior in the passband. It presents graphical and numerical tools for designing such filters, as well as a number of realizable circuit topologies for robustly implementing this class of filters. Aside from direct applications to the design of filters with asymptotically lower passband error, understanding such filters helps shed light on several unusual phenomena, such how overshoot can occur in pure RC transmission lines (with driven shielding, but no inductive elements). The paper is presented in the context of lowpass filters, but is equally applicable to band-pass and high-pass filters through standard transformations.**

I. Introduction

Filter design is concerned with maximizing attenuation in the stopband, while minimizing error in the passband. An ideal filter will transmit the signal without alteration in the passband, and attenuate the signal completely in the stop-band. In practice, such a transfer function is impossible to achieve – there will always be some error in the passband, and the attenuation in the stopband will not be complete. Many classes of approximations to this ideal have been introduced. These approximations are generally optimal, but differ in what metrics they optimize over. The Butterworth filter approximation attempts to minimize amplitude error in the passband $E(s)=1-|H(s)|$ (at the cost of phase error). The Bessel filter approximation attempts to minimize phase error, approximating a simple delay. This paper introduces a new metric for error: absolute error $E(s)=|1-H(s)|$, and describes a family of filters which achieves asymptotically better performance with this metric than traditional causal filter design.

The most common analog filter types – passive RC, Butterworth, and Bessel filters – consist of only poles. These result in transfer functions where the absolute error $E(s)=|1-H(s)|$ has a roll-off of 20dB/decade. This is fundamental to all stable, causal all-pole filters. Such a filter has a transfer function of the form:

$$H(s)=\frac{1}{1+c_1s+c_2s^2+c_3s^3+...+c_ns^n} \qquad (1)$$

For stability, $c_1\neq 0$, so the asymptotic behavior of the error at the origin can be approximated as:

$$E(s)=|1-H(s)|\approx|c_1s| \text{ as } s\to 0 \qquad (2)$$

The slope of $E(s)$ near the origin sets the asymptotic performance of the filter. I will define this as the *order of error roll-off* in the passband. In the above case, this slope is 20dB/decade, so the passband error roll-off is first order. Standard pole-zero filter types, such as the elliptic and Chebyshev combine poles with zeros to give faster crossover from passband to stopband, but still only give first order passband error roll-off. This paper discuses a class of filters which use zeros to give better asymptotic performance in the passband. These filters give coincident terms in the numerator and denominator, which cancel as $s\to 0$. The transfer functions of such filters can be written as:

$$H(s)=\frac{1+\sum_{i=1}^{k-1}c_is^i+\sum_{i=k}^{p}n_is^i}{1+\sum_{i=1}^{k-1}c_is^i+\sum_{i=k}^{q}d_is^i} \qquad (3)$$

$\sum c_is^i$ are coincident terms, while $\sum n_is^i$ are non-coincident numerator terms (if any), and $\sum d_is^i$ are non-coincident denominator terms. This results in filter with kth order error roll-off in the passband ($k\cdot 20$ dB/decade), and $(q-p)$th order filter roll-off in the stop-band. Specifically, Taylor expanding equation 3, asymptotic behavior near the origin is:

$$E(s)=|1-H(s)|\approx\left|k!\,(n_k-d_k)\,s^k\right| \text{ as } s\to 0 \qquad (4)$$

While the author has seen filters of this form accidentally used in certain types of circuits (generally without correct analysis), the author is unaware of any comprehensive treatment of the theory of these filters. This paper presents such a theory. While this paper presents the theory in the context of low-pass filters, the theory is equally applicable to other classes of filters by the standard transformations [1].

This class of filters has several unusual properties. First of all, filters with error roll-off in the passband of order greater than 1 will, fundamentally, give a region where the gain is greater than 1. This phenomenon shows up even in purely passive RC implementations of such filters. I will call this phenomenon *pseudoovershoot*. It looks very similar to traditional overshoot on a bode amplitude plot. However, in contrast to traditional overshoot, it may be caused either by a resonance, or by a zero followed by a pole. In the latter case, it will have very different properties from normal overshoot – in particular, it does not have the same associations with instability, and can happen in circuits composed of purely passive RC elements. Excessive pseudoovershoot can still cause issues – it may lead to substantial phase dispersion near crossover resulting in long-lasting high-frequency transients in the step response. This paper introduces and explains this phenomenon, and provide some strategies for managing it.

The first half of the paper explores the mathematics of this class of filters. Section III shows a simple, intuitive graphical technique for placing poles and zeros that can be used to generate filters with the first two terms coincident (or 40db/dec error roll-off in the passband). Section V derives the rules

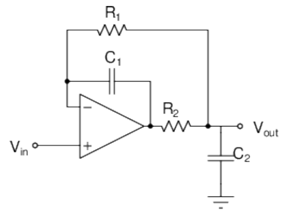

Fig. 1. A topology for a commonly used antialiasing filter or capacitive load driver. The topology is designed to isolate the operational amplifier from capacitive loading at high frequencies by providing a feedback path through C_1, while acting as a buffer at low frequencies, by providing a feedback path through R_1. This circuit exhibits 40dB/dec passband error by effectively removing the capacitive load at low frequencies using feedback through R_1. It exhibits 20dB/dec fall-off at high frequencies, by acting as a buffer (with feedback through C_1) followed by an R_2C_2 filter.

for the technique from section III, and shows that they are complete – they allow us to derive *all* filters with second order passband error roll-off, and all filters derived using it have at least 40dB/dec passband roll-off. Section VI explains a simple non-graphical technique for designing filters of this class that leverages existing filter approximation techniques. It also proves that filters of this class must have pseudoovershoot. Sections VII and VIII show two types of practical active implementations where passband error roll-off is guaranteed by topology (and therefore is insensitive to component variation). Note that this is a departure from most classic filter realizations which approximate ideal filters with imperfect component values, and so are limited by component variation. In contrast, we bound $|1 - H(s)|$ *even with component errors*. Since this also places a bound on $1-|H(s)|$, at sufficiently low frequencies, it outperforms realistic implementations of filters such as the Butterworth even on the Butterworth's figure-of-merit. Both topologies are constructed by modifying traditional filter realizations, and so virtually any filter (biquad, Sallen Key, passive, etc.) can be modified into this class of filters. For clarity, I will demonstrate this on a very simple filter. Section IX describes a passive RC circuit implementation. Section X shows how the analysis applies to the design of shielded transmission lines.

II. Related Work

Filters of this form appear in a number of places. The most common application is transmission lines with driven shields. By driving the shield with a finite bandwidth, one effectively removes the capacitance up to the frequency of the driven shield. In the case of an RC transmission line, the error of the signal on the shield scales at 20dB/decade. The error of the signal on the main conductor, then, scales at 40dB/decade.

There is a standard topology shown in figure 1 that is commonly used to drive sigma-delta ADCs [2] and other capacitive loads. This topology has 40dB/decade error in the passband. While the first reference to this topology is lost in antiquity, it commonly appears in app notes. At high frequencies, the operational amplifier acts as a buffer through C_1, and the output is filtered through R_2/C_2. At low frequencies, feedback through R_1 effectively removes R_2/C_2.

A number of speaker crossover designs claim to have this property (if implicitly). The goal is to have substantial roll-off in the stop band ($>$ 20dB/dec), while maintaining constant total signal into the woofer and tweeter: $H_{LP} + H_{HP} = 1$. If both constraints are met, the order of the error roll-off must be greater than 1. In reality, the only cross-over network I found which actually had this property was a simple LC voltage divider[1].

The concept of characterizing a polynomial transfer function by a single time constant was in the early 1960s in the development of open circuit time constants [4]. Much of the theory of this class of filters comes from matching the time constants of the numerator and denominator, and therefore follows directly from their work.

We have known since the 1950s that completely passive RC circuits could exhibit voltage gain, but this was treated as a curiosity [5]. While several such circuits exhibited 40dB/dec stop-band error, the authors did not appear to realize that those circuits had additional interesting properties.

The mathematics here corresponds directly to that of control systems with zero steady state error in response to higher-order inputs (ramps, parabolas, etc.). Indeed, circuits synthesized as feedback loops with multiple integrators in the feedback path do give higher order passband error roll-off. I do not focus on this as a synthesis technique, since for this class of systems, avoiding conditional stability[2] requires careful attention to saturation behavior. However, the mathematics developed in controls is directly applicable to this class of filters, and vica-versa.

III. Graphical Design Technique

There is a simple, graphical technique for designing filters with 40dB error roll-off in the passband. In this section, I will explain this technique, demonstrate how to use it. In section V, I will formally derive the rules, and show that they are complete – every set of poles and zeros with 40dB/decade error roll-off can be reached using the technique, and conversely, every set of poles and zeros reached will have 40dB/decade error roll-off in the passband.

The graphical synthesis technique consists of five, simple graphical operations for manipulating poles and zeros without effecting the 40db/dec roll-off of $E(s)$ in the passband:

1) Start with a system with 40dB/decade error roll-off in the passband (typically, the system with no poles or zeros – $H(s) = 1$)
2) Add (or remove) a pole-zero pair in the same location
3) Replace a real pole or zero at frequency f with a pair of singularities of the same type at $2f$ (or vica-versa).
4) Move complex conjugate pairs of singularities along circles centered on the real axis passing through the origin.
5) Slide singularities of the same type along the real axis, keeping the constraint that the sum of their reciprocals is constant.

[1]More sophisticated crossover networks have substantial additional constraints due to claimed compensation for spatial interference patterns from the physical spread of the woofer and tweeter, which appear to have more influence on filter design than constraints such as $H_{LP} + H_{HP} = 1$ [3].

[2]A conditionally stable system is one in which a reduction in loop gain (for instance, as caused by saturation) may give rise to instability.

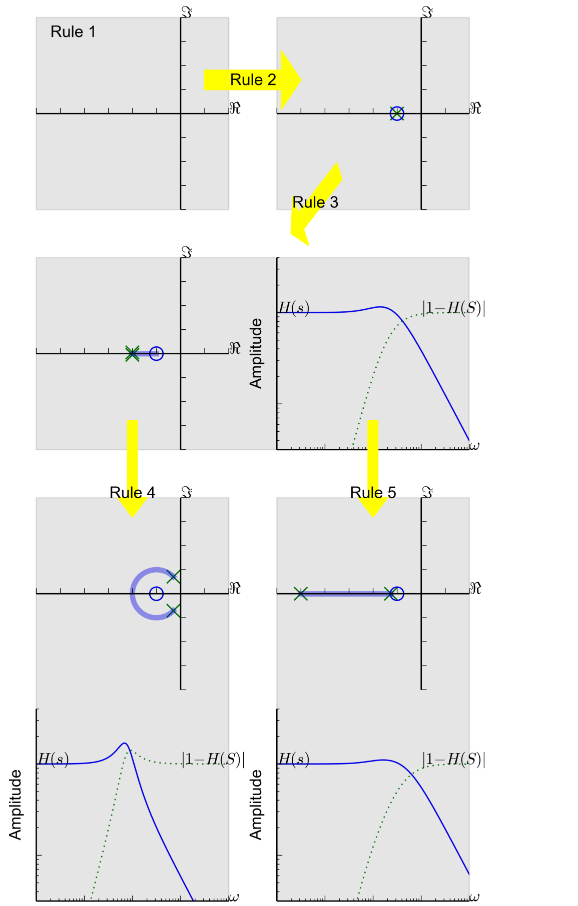

Fig. 2. A graphical illustration of the procedure in section III. By rule 1, start with $H(s) = 1$. By rule 2, place a pole-zero pair at 1 rad/sec. By rule 3, split that pole into a pair of poles at 2 rad/sec. Alternatively, by rule 4, move those poles towards the imaginary axis, or by rule 5, move those poles along the real axis. In the bode plots, the solid line shows the frequency response of the system $|H(s)|$, while the dotted line shows the frequency response of the passband error, $E(s) = |1 - H(s)|$.

IV. Worked Example

I give an example of how this is used in practice in figure 2. This gives a filter with one zero and two poles, giving 20dB/dec roll-off at high frequencies. If the poles are coincident on the real axis, the zero occurs at half of the frequency of the poles, resulting in a region with a small amount (15%) of pseudoovershoot in the crossover region. If less pseudoovershoot is desired, spreading the poles by rule 5 reduces it, at the cost of a longer transition region from pass-band to stop-band. Alternatively, if the crossover is too slow, applying rule 4 splits the poles onto the complex plane and moves them towards the origin, replacing pole-zero pseudoovershoot with a greater amount of normal resonant peaking. The complete set of possible pole placements for a system with a single zero and two poles is shown in figure 4 (this is the minimal case – most filters will apply rules many more times, leading to more poles and zeros).

V. Derivation of Rules

Theorem: Any realizable rational transfer function can be characterized by two, real time constants – one associated with the numerator, and one with the denominator. Those time constants are equal to the sum of the time constants of the individual poles or zeros. For 40dB/decade passband error roll-off, the time constant of the numerator must equal the time constant of the denominator.

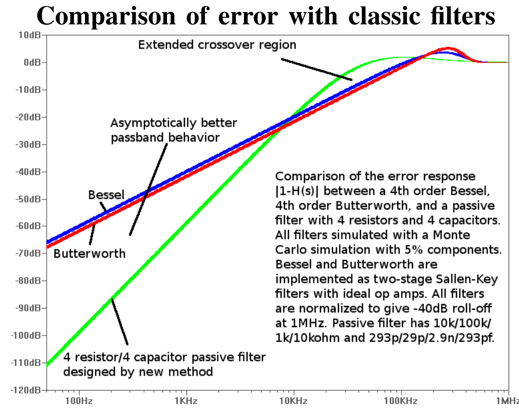

Fig. 3. A comparison between classic filters and a simple passive filter designed with the new methodology (topology from Fig. 8). This comparison pits complex, optimized 4^{th} order active filters (2 op-amps/8 passives) against just four passives. The passive topology gives low pseudoovershoot ($< 2dB$) and wide crossover. An active implementation could shrink the crossover region by increasing pseudoovershoot. The Monte Carlo simulation demonstrates the filter error rolls of at 40dB/dec even with imperfect components. In contrast, realistic realizations of a filter like the Butterworth would only have a limited region where $1 - |H(s)|$ rolled off with the Butterworth's ideal slope (note: figure shows $|1 - H(s)|$, not $1 - |H(s)|$). Past that point, the Butterworth would limit to 20dB/dec, or even 0dB/dec, depending on realization. The low apparent spread in all plots is due to the large 130dB vertical scale.

Take a transfer function of the form:

$$H(s) = \frac{1 + \sum_{i=1}^{n} c_i s^i}{1 + \sum_{i=1}^{n} d_i s^i} \quad (5)$$

The Taylor expansion of $H(s)$ around the origin is:

$$H(s) \approx 1 + (c_1 - d_1)s + (c_2 - d_2 - c_1 d_1 + d_1^2)s^2 + \ldots \quad (6)$$

For $E(s) = |1 - H(s)|$ to converge to 0 at the origin at 40dB/dec, $c_1 = d_1$. Define c_1 as the time constant of the numerator, and d_1 as the time constant of the denominator.

For 40dB/dec passband error roll-off, it is sufficient and necessary that the s term of be coincident between the numerator and denominator of the transfer function. This term is *the time constant of the polynomial*. Notice that this time constant is additive:

$$(1 + \alpha s + \ldots) \cdot (1 + \beta s + \ldots) \approx (1 + (\alpha + \beta)s) + \ldots \quad (7)$$

Therefore, the time constant of a polynomial is equal to the sum of the time constants of the zeros of that polynomial. For 40dB/dec passband error roll-off, the sum of the time constants of the poles minus the sum of the time constants of the zeros must equal zero. Since the imaginary parts must, by necessity, cancel, it is sufficient to phrase this as a constraint on the real parts of the time constants:

$$\sum \Re\left(\frac{1}{z_i}\right) = \sum \Re\left(\frac{1}{p_i}\right) \quad (8)$$

Singularities may also be moved along the real axis, as long as the time constant $\sum \frac{1}{\omega}$ remains constant. Hence, rule 5

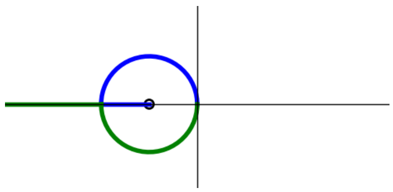

Fig. 4. Shown is the set of pole positions for a pair of poles which give equivalent total time constant. The pair of poles gives the same contribution as a single pole placed at the center of the circle if they are both either on the circle, or if they are both on the real axis with a fixed harmonic mean equal to the position where the circle intersects the real axis.

follows. Similarly, a singularity may be split into two with the same total time constant. Hence, rule 3 follows.

The set of points which satisfies $\Re(\tau) = c$ is a vertical line. By the conformal mapping from time to frequency, the set of points which satisfies $\Re\left(\frac{1}{\omega}\right) = c$ is defined by a circle passing through zero and $\frac{1}{c}$. Singularities may be moved along this circle without effecting the overall time constant. This gives rise to rule 4. The complete set of equivalent positions for a pair of poles is, again, shown in figure 4.

This set of rules is complete – the rules can transform any transfer function with 40dB/decade passband error roll-off into $H(s) = 1$. Completeness can be shown by construction. First, move all singularities onto the real axis by rule 4. Then, merge pairs of singularities with rule 3. This leaves a single pole and a single zero. Since the rules preserved 40dB/decade passband error roll-off, that pole and that zero must be coincident. Remove them by rule 2.

VI. Direct Synthesis Technique

Filters of this form may also be created through direct synthesis from a chosen error function. In this methodology, we drop the absolute value from the definition of $E(s)$, and use $E_n(s) = 1 - H(s)$. The error function $E_n(s)$ can be any high-pass filter designed by traditional filter approximation techniques. Then, the filter transfer function can be explicitly calculated as $H(s) = 1 - E_n(s)$.

In this case, if $E_n(s)$ is a kth order filter, it is easy to show $H(s)$ has k coincident terms. Given the passband error function:

$$E_n(s) = \frac{\sum_{i=k}^{m} d_i s^i}{1 + \sum_{i=1}^{p} c_i s^i} \tag{9}$$

The transfer is:

$$H(s) = 1 - \frac{\sum_{i=k} d_i s^i}{1 + \sum_i c_i s^i} = \frac{1 + \sum_i c_i s^i - \sum_{i=k} d_i s^i}{1 + \sum_i c_i s^i} \tag{10}$$

Notice that the locations of the poles (but not zeros) is identical between the stopband error function $E_n(s)$ and the filter transfer function $H(s)$.

Unless the high-pass function $E_n(s)$ also has coincident terms (which is sometimes a complex design constraint), the stopband roll-off of $H(s)$ will only be 20dB/decade. If this is not sufficient, filters can be cascaded to give higher-order roll-off in the stop-band. Attenuation in the stopband is multiplicative, while error in the passband is additive. As a result, cascading stages improves the order of stop-band roll-off, but maintains the same order passband error roll-off (only adding a constant offset). Such a filter design is shown in figure 6. Of course, pseudoovershoot typically becomes worse with cascaded filters.

The direct synthesis methodology directly and intuitively shows why this class of filters must have pseudoovershoot – whenever $90^\circ < \angle E_n(s) < 270^\circ$, $E_n(s)$ will sum constructively with the signal, giving $|H(s)| > 1$. Since, by definition, $E_n(s)$ is a high-pass filter with >20dB roll-off and unity gain at high frequencies, if $E_n(s)$ is minimum phase, the filter must pass through a region where the phase enters this range. We can use traditional filter design techniques to minimize the frequency range of this region, minimize the amplitude of $E_n(s)$ in this region, but without right half plane singularities, we fundamentally cannot avoid it.

The core advantage of direct synthesis is that it allows the designer to leverage traditional filter approximations, such as the Butterworth [6], as well as directly apply more recent computational optimization techniques, such as vector fitting [7].

VII. Filtered Signal Ground Topology

This section describes a technique for cascading conventional low-pass filters to give an active topology for implementing filters with higher order error roll-off in the passband. The circuit guarantees coincident terms by topology, independent of component values or component matching. By this technique, any pair of active, low-pass filters $H_1(s)$ and $H_2(s)$, with nth order and mth order error roll-off in the passband can be converted into a single filter with $(n+m)$th order error roll-off in the passband.

Since voltages are differential, any circuit implementation of a filter transfer function $H_1(s)$ is implicitly ground-referenced: $V_{out}(s) = V_{in}H_1(s) + V_{ref}(1 - H_1(s))$ where V_{ref} is typically circuit ground. By replacing ground with a node that, at low frequencies, follows V_{in}, we can create filters with greater attenuation in the passband. Let $V_{ref} = H_2(s)V_{in}$. Then,

$$V_{out} = V_{in}H_1(s) + H_2(s)V_{in}(1 - H_1(s)) \tag{11}$$

Then,

$$H(S) = \frac{V_{out}}{V_{in}} = H_1(s) + H_2(s) - H_1(s)H_2(s) \tag{12}$$

Let the Taylor approximation around DC of the transfer functions be $H_1(s) \approx 1 + cs^n + ...$ and $H_2(s) \approx 1 + ds^m + ...$ Then:

$$H(S) = \frac{V_{out}}{V_{in}} = 1 + (c + d)s^{n+m} + ... \tag{13}$$

We can cascade an arbitrary number of filters this way to generate arbitrarily good passband error roll-off. The high-pass behavior is dominated by the lowest-order filter. A sample filter of this type is shown in figure 5.

VIII. Inverse Highpass Topology

As implied in section VI, a second way to design this class of filters is by subtracting the output of a highpass filter from the signal. As with the previous topologies, the passband error roll-off is independent of component values. The stopband behavior, however, is dependent on gain match between the

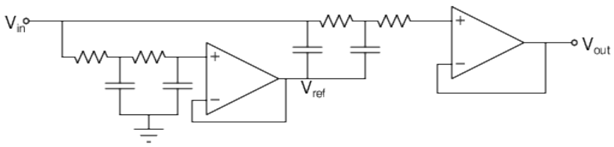

Fig. 5. A cascade of two low-pass filters designed to give 40db/decade passband error roll-off. Here, V_{ref} follows the signal at low-frequencies. V_{out} is filtered relative to V_{ref}, and so has asymptotically better performance in the pass-band.

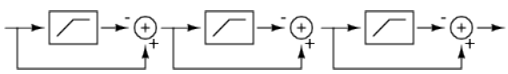

Fig. 6. A lowpass filter consisting of a cascade of $1 - H_{HP}(s)$.

signal and the highpassed signal, as well as on the CMRR of the subtractor. A block diagram of this is shown in figure 6. A sample circuit is shown in 7.

IX. Passive Implementation

Figure 8 shows a passive RC circuit topology of a filter which guarantees 40dB roll-off in both in the stopband and of the passband error. This circuit is simulated in figure 3. For simplicity, I present the case where all resistors are equal and all capacitors are equal. In this case, the transfer function is[3]:

$$H(s) = \frac{1 + 8RCs}{1 + 8RCs + 8R^2C^2s^2 + R^3C^3s^3} \tag{14}$$

Although the circuit is purely passive, as shown in section VI, this filter still has a small amount of voltage gain at crossover (0.84dB) (indeed, this circuit topology was first introduced in [5]). A grid of this form can be scaled arbitrarily in either direction. Scaling the circuit in the horizontal direction increases stop-band roll-off (giving nth order roll-off, where n is the number of horizontal stages). Scaling the circuit in the vertical direction improves passband error roll-off (giving mth order roll-off, where m is the number of vertical stages). Similar circuits could also be constructed as LC or RLC filters.

[3]As a point of curiosity, in all cases where the number of columns equals the number of rows, the denominator of the transfer function appears to be a symmetric polynomial. I could not prove this result, but I was able to demonstrate it for this topology up to the case of 5×5.

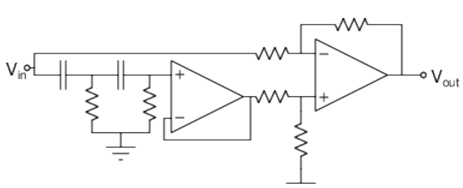

Fig. 7. A lowpass filter with 40dB/dec passband error roll-off, and 20dB/dec stopband roll-off. Passing the signal through multiple such filters would give better stopband roll-off, while maintaining 40dB/dec passband error roll-off. The passband error roll-off is insensitive to component values, but the stopband behavior is limited by the CMRR of the difference amplifier.

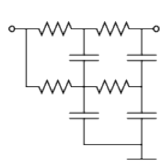

Fig. 8. A passive filter topology that gives 40dB attenuation in the stopband, and 40dB error attenuation in the passband.

TABLE I

Normalized rise time to a step input for an RC line with 0, 1, 2, or 3 shields. In the first case, the simulation assumes constant capacitance between traces. In the second, it assumes constant total capacitance to outer shield.

# Shields	0	1	2	3
70% Rise (C constant)	1	0.31	0.15	0.090
70% Rise (C scaled)	1.	0.61	0.45	0.36
100% Rise (C constant)	∞	0.60	0.28	0.16
100% Rise (C scaled)	∞	1.20	0.83	0.65
Overshoot	0	9.6%	12.8%	14.2%

X. Distributed Transmission Line

The circuit in section IX scales to distributed implementation, and so offers some insight into to the design and analysis of transmission lines with driven shields. If we extend the filter from section IX horizontally, for a 20dB roll-off case, we are left with an RC transmission line. With 40dB roll-off, we are left with an RC transmission line with a driven shield. With 60dB roll-off, we are left with a transmission line with two driven shields. We can immediately see that even a simple, unloaded RC transmission line with a driven shield must have a small amount of overshoot, even with no inductance. Using a finite approximation of the transmission line of 50 RC segments, the overshoot (as well as rise time) of such a transmission line is shown in figure I. The analysis for driven LC or RLC transmission lines is similar.

XI. Conclusion

This paper presented a class of filters that can give arbitrarily good response in the passband, as well as practical techniques for both designing and implementing such filters. Under reasonable assumptions, this class of filters must have a region of gain greater than one near crossover, even with completely passive RC implementations. A key downside of this class of filters is the relatively wide crossover region. As a result, they lend themselves well to applications where the stopband is far from the passband, such as before an oversampling ADC.

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